



COLD ATOMS STIRRING

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January - April 2013

Abstract

This report consists in a theoretical study of the effect of quantum stirring of fermionic systems in one dimension. “Stirring” consists in dragging the fluid by means of a laser moving through the system at constant speed. The laser interacts differently with the system depending on its phase and the strength of the interactions. Stirring can therefore be used as a probe of the collective properties of the system.

Fermi gases with different internal symmetries are considered. First the case of fermions with no internal degree of freedom is studied, along with a presentation of the formalism. The stirring of a two-component fermionic gas exhibiting an $SU(2)$ symmetry is investigated in a second place.

Résumé

Ce rapport consiste en une étude théorique de l’effet du touillage quantique sur des systèmes fermioniques en une dimension. On entend par « touillage » l’opération consistant à entraîner le fluide fermionique à l’aide d’un laser avançant à travers le système à vitesse constante. Le laser interagit différemment avec le système selon sa phase et la force de ses interactions. Le touillage peut donc être utilisé pour sonder les propriétés collectives du système.

Des gaz de Fermi à différentes symétries internes sont considérés. En premier lieu, le cas de fermions sans degré de liberté interne est étudié, ce qui permet d’exposer le formalisme développé. Le touillage d’un gaz à deux composantes présentant une symétrie $SU(2)$ est envisagé dans un second temps.

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Introduction

In 1995, the team of Cornell, Wieman and Ketterle opened a new chapter in our understanding of the quantum world [1]. By trapping ^{87}Rb atoms in laser-generated potentials, they were able to realize experimentally the bosonic condensates predicted seventy years before by Bose and Einstein [2, 3]. Quantum degeneracy in fermionic gases, inherently more difficult to cool down, took four more years to be reached, using ^{40}K atoms [4].

These breakthroughs have since motivated a huge effort both by theorists and experimentalists. The parameters of a cold atoms experiment – the nature of the atoms, their internal degrees of freedom, the interactions, the lattice and its geometry or dimensionality – can now be chosen and tuned to such an extent that it becomes increasingly possible to simulate any condensed matter system. Quantum gases become laboratories probing theories emanating from various fields of physics, from Mott insulators [5] to the Higgs mechanism [6]. Tunable, defectless, and providing an access to undreamed-of observables, cold atoms systems have opened new ways to investigate quantum properties of matter.

A particular topic of investigation in cold atoms is the effect of dimensionality: by acting on the trapping potential, one may create various systems approaching a one or two-dimensional behavior. One-dimensional systems, on which this work focuses, can notably describe “cigar-shaped” or annular geometries [7]. In one dimension, the effects of interactions and quantum fluctuations are drastically enhanced; the behavior of these systems is therefore of great interest [8].

Laudau’s Fermi liquid theory [9], describing interactions in “normal” metals, breaks down at one dimension. Indeed, the elementary excitations of the system can no longer be seen as electrons “dressed-up” by interactions. Instead, these excitations are necessarily collective, as in the paradigmatic Luttinger liquid picture discussed further in this work. Because of the “collectivization” of elementary excitations, one-dimensional fermionic systems may prove very difficult to be analysed in terms of their fermionic constituents. However, when at a quantum critical point, one may use the technique of bosonization stemming from Conformal Field Theory (CFT) [10] to simplify the problem by mapping the system of fermions to a system of bosons. Because of the splitting of their constituents into collective excitations, one-dimensional systems may have a vast spectrum of “exotic” phases where charge fractionalization related effects are common. The classification of these phases and their realization is still the object of active research; proposals for realizing 1D-states in cold atoms experiments include the observation of Mott insulators [5], fractional Quantum Hall effect [11] and spin chains [12, 13].

In order to investigate the phase of a cold atom system, one may want to probe its elementary excitations by applying a localized perturbation moving at a constant speed by stirring a laser through the system. The motion of the laser gives an orientation to the system, and breaks a symmetry between atoms moving to the right and atoms moving to the left. One therefore expects the creation of a current. This work analyzes the response of a one-dimensional fermionic system to quantum stirring using the analytical methods discussed earlier.

In general, atoms trapped on a lattice have several internal degrees of freedom. They correspond *e.g.* to different electronic orbital configurations, hyperfine spin states, or different atom species in atomic mixtures. If the interactions between atoms and the interactions with the lattice are independent of these internal components, there is a symmetry in the system. For example, atoms trapped with two degenerate electronic orbitals will exhibit a $SU(4)$ symmetry (taking the spin of the electrons $\sigma = \uparrow, \downarrow$ into account). High symmetries allow atoms to pair into exotic configurations, like high-spin Cooper pairs [14] or molecular

bound states of more than two atoms [15]. These rich behaviors explain the efforts made towards experimentation of these systems. Having realized a system with high symmetry, one could in principle study any system of lower symmetries by reducing the number of populated degrees of freedom, *e.g.* by optical pumping or selective evaporation.

However, the realization of these large symmetries present numerous experimental challenges, as it is difficult to trap the atoms without breaking the symmetry between their degrees of freedom. The cooling of fermionic isotopes of alkaline-earth-metal atoms like Ytterbium isotopes [16] gives promising results, because the nuclear spin of these atoms in the ground state decouple from collision processes. In 2010, Takahashi *et al.* reported the realization of an impressive $SU(6) \times SU(2)$ symmetry in a mixture of Ytterbium isotopes [17].

Consider the case of a $SU(2)$ symmetry realized by atoms trapped on a lattice with two internal degrees of freedom $\sigma = \uparrow, \downarrow$. If the lattice wells are deep enough, one can treat the fermions in the tight-binding approximation, in which only tunnel motion is considered. At low temperature, only the lowest energy band is occupied and the Hubbard model accurately describes the system [18]:

$$H = - \sum_{i,j,\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}.$$

The first part describes hopping between sites: t_{ij} is the hopping amplitude between site i and site j , and can be neglected if these sites aren't nearest neighbors; $c_{j\sigma}$ annihilates an atom at site j and $c_{i\sigma}^\dagger$ creates an atom at site i . The second part modelizes same-site interactions between atoms: $n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma}$ is the density of specie σ at site i . The interaction intensity U is proportional to the diffusion length.

As will be reminded in Section 2.2, this $SU(2)$ Hubbard model presents the interesting phenomenon of *spin-charge separation*: the fermionic excitations split into two decoupled elementary excitations, one in the charge degrees of freedom (density fluctuations) and one in the spin sector (spin waves). By varying the strength and the sign of the interaction intensity U relative to the hopping amplitude t , one obtains a quantum phase transition between a Mott insulating phase and a metallic phase [19]. One can foresee that systems with larger symmetries will exhibit an even richer spectrum of phases.

This work focuses on the response of two simple models to stirring. As a warm-up, a system of spinless fermions in interaction is first considered; the fact that this system belongs to the Luttinger universality class allows the results to be compared to a paper on quantum stirring with bosons [20]. The stirring of the spin-1/2 case discussed above is then investigated.

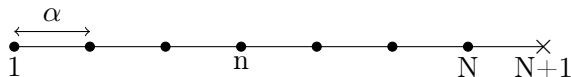
1 Spinless fermions

In this section, we consider a system of fermions with no internal degree of freedom trapped on a one-dimensional lattice. The properties of the free model are first investigated in Subsection 1.1. In Subsection 1.2, we proceed by adding interactions in the system, and expose some properties of the Luttinger model we obtain. Finally, Subsection 1.3 presents the response of this system to stirring.

1.1 Free fermions

We first investigate the properties of the free model, where fermions hop freely on the lattice. This Hamiltonian is exactly solvable and will be the Gaussian fixed point of our perturbation theory. A low-energy linearization of the Hamiltonian spectrum will allow the emergence of a conformal symmetry, with two distinct excitations of opposed chiralities, “right” and “left”-movers. This will allow us to use Conformal Field Theory to express two-point correlation functions of the model and use bosonization to obtain an equivalent picture of the system in terms of free bosons.

We consider free fermions hopping on a 1D chain.



The tight-binding Hamiltonian describing the system is given by:

$$\mathcal{H}_0 = -t \sum_{n=1}^N c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n \quad (1.1)$$

where the fermionic creation and annihilation operators satisfy the usual anticommutation relations $\{c_n, c_{n'}\} = \{c_n^\dagger, c_{n'}^\dagger\} = 0$ and $\{c_n, c_{n'}^\dagger\} = \delta_{n,n'}$, and we consider antiperiodic boundary conditions: $c_{N+1} = -c_1$. The number of particles $\mathcal{N} = \sum_{n=1}^N c_n^\dagger c_n$ is conserved as it commutes with \mathcal{H} ; it is the conserved charge of the $U(1)$ symmetry associated to the transformations $c_n \rightarrow e^{i\theta} c_n$ which leave the Hamiltonian invariant. This Hamiltonian is diagonal in Fourier space: $\mathcal{H}_0 = \sum_k \epsilon(k) c_k^\dagger c_k$ where the operators in momentum space $c_k = \frac{1}{\sqrt{N}} \sum_n c_n e^{ikna}$ still satisfy canonical anticommutation relations (by unitarity of Fourier transform). The single-particle spectrum given by $\epsilon(k) = -2t \cos(k\alpha)$ in the Brillouin zone $k \in [-\pi/\alpha, \pi/\alpha]$ is plotted on Figure 1.1.

At zero temperature, the fermions fill the energy levels up to the Fermi energy. Low energy excitations of the system are located close to this Fermi energy; we will therefore linearize the dispersion relation close to the two Fermi points $k = \pm k_F$. The linearized spectrum includes two distinct branches, distinguishing “left movers” for which the dispersion writes $\epsilon(p = k + k_F) - \epsilon_F = -v_F p$ from “right movers” for which it writes $\epsilon(p = k - k_F) - \epsilon_F = v_F p$, where the Fermi velocity $v_F = 2t\alpha \sin(k_F \alpha)$. We shall denote fermionic operators in the left and right branches by $L_p = c_{-k_F+p}$ and $R_p = c_{k_F+p}$ respectively. The ground state of the system is the Fermi Sea, which may be written $|FS\rangle = \prod_{p<0} R_p^\dagger \prod_{q>0} L_q^\dagger |0\rangle$. Hamiltonian (1.1) reads

$$\mathcal{H}_0 = \sum_{|p|<\Lambda} v_F p \left(: R_p^\dagger R_p : - : L_p^\dagger L_p : \right) \quad (1.2)$$

where a cut-off $\Lambda \sim \alpha^{-1} \ll k_F$ has been introduced in order to account for the breakdown of validity of the model at short wavelengths (in the following we use $\hbar = 1$; therefore Λ is also a

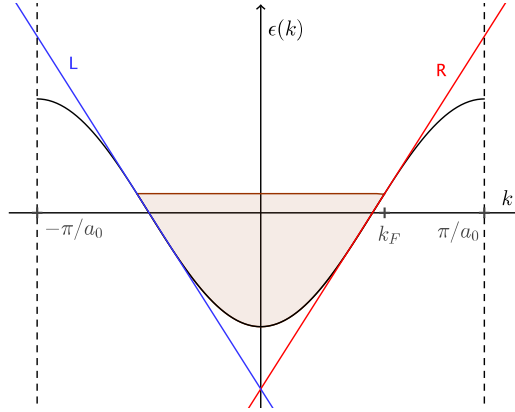


Figure 1.1 – Dispersion relation for the 1D tight-binding Hamiltonian. When linearized close to the Fermi points, the spectrum exhibits two distinct branches L and R .

cut-off in energy). The terms have been normal ordered to cancel out their mean value in the ground-state, granted that for any operator $\hat{\mathcal{O}}$, one has $:\hat{\mathcal{O}}:=\hat{\mathcal{O}}-\langle FS|\hat{\mathcal{O}}|FS\rangle$.

By inverse Fourier transform, the low energy annihilation operator on the lattice is given by

$$c_n \approx \sqrt{\alpha} \left[R(x)e^{ik_F x} + L(x)e^{-ik_F x} \right] \Big|_{x=n\alpha} \quad (1.3)$$

where we do not restrict ourselves to discrete values of x in the continuum limit: $\alpha \rightarrow 0$ at fixed $L = N\alpha$, $c_n \rightarrow \sqrt{\alpha}c_n(x = n\alpha)$, $\sum_n \alpha \rightarrow \int_0^L dx$ (replacing the discrete δ by a Dirac function in the anticommutation relations). This limit is justified by the fact that the large-wavelength physics is not sensitive to the lattice spacing. Injecting this expression in the expression of the Hamiltonian (1.2), we obtain

$$\mathcal{H}_0 = -iv_F \int_0^L dx \left[:R^\dagger(x)\partial_x R(x): - :L^\dagger(x)\partial_x L(x): \right]. \quad (1.4)$$

By Legendre transformation, Hamiltonian (1.4) is associated to the massless Dirac Lagrangian in 1+1 dimensions $\mathcal{L} = iv_F \bar{\Psi} \gamma^\mu \partial_\mu \Psi$, where Ψ is the spinor (R, L) . It decomposes into two commuting chiral parts \mathcal{H}_R and \mathcal{H}_L . The global $U(1)$ symmetry that already preserved the lattice Hamiltonian (1.1) becomes a subgroup of a larger emergent $U(1)_R \times U(1)_L$ symmetry associated to the transformations $R_n, L_m \rightarrow e^{i\theta} R_n, e^{i\phi} L_m$, accounting for the conservation of the charge both in the left and right sectors. The associated conserved currents are $J_R = :R^\dagger R:$ and $J_L = :L^\dagger L:$. Conformal invariance in 1+1 dimensions stems from these chiral symmetries, which puts strong constraints on the theory allows us to easily compute the correlators of the theory (Green's functions) analytically.

These correlators are summarized in Appendix A.1, along with some Operator Product Expansions (OPEs) of the fermionic fields. They show that the square of the currents express as $:J_L^2: = :L^\dagger \partial L:$ and $:J_R^2: = :R^\dagger \partial R:$. We may thus express Hamiltonian (1.4) in terms of the chiral currents:

$$\mathcal{H}_0 = \pi v_F \int_0^L dx \left[:J_L^2: + :J_R^2: \right]. \quad (1.5)$$

As shown in appendix A, the Hamiltonian of free massless bosons in 1+1 dimensions also takes this form when expressed in terms of its conserved currents. This is no coincidence, as our system may be mapped to a system of bosonic fields using the nonlinear transformation:

$$R = \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\phi_R}, \quad L = \frac{1}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\phi_L}. \quad (1.6)$$

The bosonic fields ϕ_R, ϕ_L satisfy the commutation relations of the free bosonic theory $[\phi_R(x), \phi_R(x')] = \frac{i}{4} \text{sgn}(x - x')$, $[\phi_L(x), \phi_L(x')] = -\frac{i}{4} \text{sgn}(x - x')$ and $[\phi_R(x), \phi_L(x')] = \frac{i}{4}$. In terms of the bosons, the chiral currents express as $J_R = :\partial_x \phi_R: / \sqrt{\pi}$ and $J_L = :\partial_x \phi_L: / \sqrt{\pi}$. Reinjecting these expressions (1.5), one recovers the Hamiltonian of free massless bosons in terms of the total field $\Phi = \phi_L + \phi_R$ and its dual $\Theta = \phi_L - \phi_R$:

$$\mathcal{H}_0 = \frac{v_F}{2} \int dx \left[:(\partial_x \Phi)^2: + :(\partial_x \Theta)^2: \right]. \quad (1.7)$$

One verifies that every two-point correlation functions, OPEs and (anti-)commutation relations coincide both in the fermionic and in the bosonic theories. Bosonization thus provides a dictionary, summarized in Appendix A.1, which allows us to switch between the fermionic and the bosonic pictures, which will be very useful. Indeed, bosonization often provides a much better basis than the original fermionic fields to study the low energy behavior of interacting fermionic systems.

1.2 Interacting fermions

We proceed by adding interactions in the model. Since two fermions cannot be on the same site, adding single-site interactions would only redefine the chemical potential, leaving the physics unchanged. Hence the simplest interaction term we could add to the free Hamiltonian \mathcal{H}_0 is nearest-neighbours interactions:

$$\mathcal{H} = \underbrace{-t \sum c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j}_{\mathcal{H}_0} + \underbrace{V \sum n_j n_{j+1}}_{\mathcal{H}_{\text{int}}} \quad (1.8)$$

where $n_j = c_j^\dagger c_j$ is the number operator on site j . The energy cost of having fermions on two neighboring sites is V , hence the interaction is repulsive for $V > 0$ and attractive for $V < 0$.

We place ourselves in the weak-coupling limit $|V| \ll t$, in which we expect that the bosonization approach developed in the interactionless theory is still meaningful. We now proceed to express the interaction in the continuum limit. Using the expression of the lattice operator in the continuum limit (1.3), the number operator expresses as

$$\frac{n_j}{\alpha} = J_R(x) + J_L(x) + e^{2ik_F x} L^\dagger R + e^{-2ik_F x} R^\dagger L \Big|_{x=n\alpha} \quad (1.9)$$

in terms of the chiral currents $J_R = :R^\dagger R:$ and $J_L = :L^\dagger L:$.

Using Wick's theorem, Taylor expansion and the two-point correlation functions of the fermionic fields (A.2), the product $n_j n_{j+1}$ expresses as

$$\begin{aligned} \frac{n_j n_{j+1}}{\alpha^2} = & \left(:R^\dagger \partial_x R: - :L^\dagger \partial_x L: \right) \frac{1 - \cos 2k_F \alpha}{i\pi} + 2(1 - \cos 2k_F \alpha) J_L J_R \\ & - \frac{\sin 2k_F \alpha}{\pi \alpha} (J_L + J_R) + \underbrace{\text{Cst.}}_{\text{constant}} + \underbrace{\text{Tot.}}_{\text{total derivative}} + \text{Irr.} + \text{Osc.} \end{aligned} \quad (1.10)$$

The second line of Equation (1.10) will be dropped out. Indeed,

- The total charge $J_L + J_R$ is conserved, $U(1)$ being still a symmetry of Hamiltonian (1.8), hence the first term may be absorbed in the chemical potential.
- The term Irr. containing higher-order derivatives is irrelevant in the sense of the renormalisation group approach [21]. This means its scaling dimension is greater than the dimension of spacetime $d = 2$, which implies it will have no effect on the low-energy physics of the system.

- Osc. are contributions proportional to $e^{\pm 2ik_F x}$ or $e^{\pm 4ik_F x}$; they oscillate quickly and their integral will average to zero. However, if we are at half-filling i.e. $k_F = \pi/2\alpha$, the terms proportional to $e^{\pm 4ik_F x}$ no longer oscillate. Nevertheless, in this case these terms can be shown to be irrelevant.

The first term in Eq. (1.10) is proportional to the low-energy limit of \mathcal{H}_0 presented in Equation (1.4). It can be absorbed in the free Hamiltonian by redefining v_F : the interactions renormalize the Fermi velocity. We are left with $\mathcal{H} = -iv'_F \int_0^L dx [:R^\dagger(x)\partial_x R(x) : - :L^\dagger(x)\partial_x L(x) :] + 2\pi g \int_0^L dx J_L J_R$ where the renormalized Fermi velocity $v'_F = v_F + g$ and the coupling $g = \frac{\alpha V}{\pi}(1 - \cos 2k_F \alpha)$. This expression can be rewritten using the expression of \mathcal{H}_0 in terms of the currents (Eq. (1.5)):

$$\mathcal{H} = \pi v'_F \int_0^L dx [:J_R^2 : + :J_L^2 :] + 2\pi g \int_0^L dx J_L J_R. \quad (1.11)$$

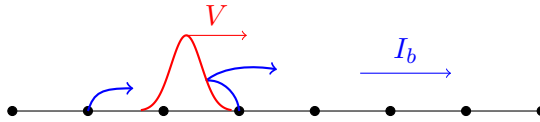
This Hamiltonian describes a Luttinger liquid. It is straightforward to apply bosonization by using the bosonic expressions of the chiral currents $J_R = :\partial_x \phi_R : / \sqrt{\pi}$ and $J_L = :\partial_x \phi_L : / \sqrt{\pi}$. The Luttinger liquid will be written in a canonical way in terms of the total bosonic field $\Phi = \phi_L + \phi_R$ and its dual field $\Theta = \phi_L - \phi_R$:

$$\mathcal{H} = \frac{v}{2} \int dx \left[\frac{1}{K} :(\partial_x \Phi)^2 : + K :(\partial_x \Theta)^2 : \right] \quad (1.12)$$

where $v = v'_F \sqrt{1 - (g/v'_F)^2} = v_F + g + \dots$ is the speed of the bosonic excitations and $K = \sqrt{\frac{1-g/v'_F}{1+g/v'_F}} = 1 - \frac{g}{v_F} + \dots$ is the Luttinger parameter parametrizing the interactions, the reason for the expansions in g/v_F being that these expressions are only valid *a priori* in the weak-coupling limit. The expressions of K and g show that $K - 1$ has the opposite sign with respect to V . The interactions are therefore attractive if $K > 1$ and repulsive if $K < 1$. As we expect, when $K = 1$ we recover the free bosonic Hamiltonian of Equation (1.7): $\mathcal{H} = \frac{v}{2} \int dx :(\partial_x \Phi)^2 : + :(\partial_x \Theta)^2 :$.

The canonical transformation $\bar{\Phi} = \Phi/\sqrt{K}$, $\bar{\Theta} = \sqrt{K}\Theta$ maps the Luttinger Hamiltonian (1.12) to the free Hamiltonian (1.7). Hence the model is critical and the elementary excitations are massless bosonic excitations.

1.3 Response to stirring



We will now investigate how the system responds to quantum stirring. Stirring consists in dragging the fluid with a laser moving through the system at a constant speed V . The laser creates a localized potential moving forward $U(x, t) = U(x - Vt)$ which couples to the density of fermions $n(x) = c^\dagger(x)c(x)$. An explicitly time-dependent term

$$\delta\mathcal{H}(t) = \int dx U(x, t)n(x) \quad (1.13)$$

is thus added to the Hamiltonian of the system (1.12)

$$\mathcal{H}_0 = \frac{v}{2} \int dx \left(\frac{1}{K} :(\partial_x \Phi)^2 : + K :(\partial_x \Theta)^2 : \right). \quad (1.14)$$

This moving potential is expected to give momentum to the fermions in the stirring direction. This would change some of the fermions moving to the left to fermions moving to the right, through processes transferring momentum $2k_F$, as pictured in Figure 1.2. Stirring gives a privileged direction to the hopping of fermions, breaking the $U(1)_R \times U(1)_L$ symmetry.

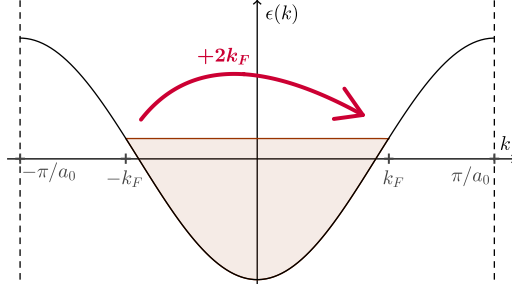


Figure 1.2 – Backscattering processes in momentum space.

The “backscattering” of left-movers to right-movers gives rise to a density current, called the backscattering current. We expect the relation between the current and the stirring velocity to depend on the strength of the interactions: by measuring this current, one could thus probe collective properties of the system. The backscattering current is naturally defined as the time derivative of the difference between the number of right-movers and left-movers $I_b = d(N_R - N_L)/dt$.

The question of the boundary conditions is of importance in an experimental setup. In an annular trap, where atoms are aligned on a ring, the backscattering current will give rise to a total angular momentum. A different behavior is expected at short times $t \ll L/V$, before the laser went round the circle, and when $t \gg L/V$, when the laser went round the system many times and the fermions are sensitive to the boundary conditions. We will in the following only investigate the case of short times, when the boundary conditions of the system have no importance to the stirring. In a cigar-shaped trap, where the currents are aligned along one axis, the current gives rise to a total momentum, and in our short-timed approach, the system behaves like an infinite system, as the current doesn’t reach the boundaries of the system.

We may assume that the interaction between the laser and the system occurs on a length that is of the order of magnitude of the lattice spacing, so that in our low-energy model, we modelize the interaction by an infinitely localized time-dependent potential $U(x, t) = U_0 \delta(x - Vt)$, which couples to the density of fermions $n(x) = c^\dagger(x)c(x)$.

The expression of the fermionic density in the continuum limit is given by Equation (1.9):

$$n(x) = J_R(x) + J_L(x) + \underbrace{e^{2ik_F x} L^\dagger R + e^{-2ik_F x} R^\dagger L}_{\text{changes } R \leftrightarrow L}.$$

As we can see, the Hamiltonian describing the perturbation $\delta\mathcal{H} = U_0 n(x = Vt)$ contains processes changing left-movers into rights-movers and vice versa.

In its bosonized version, density (1.9) becomes

$$n(x) = \frac{1}{\sqrt{\pi}} \partial_x \Phi - \frac{1}{\pi\alpha} \sin\left(\sqrt{4\pi}\Phi + 2k_F x\right),$$

hence

$$\delta\mathcal{H}(t) = U_0 \left[\frac{1}{\sqrt{\pi}} \partial_x \Phi(Vt) - \frac{1}{\pi\alpha} \sin\left(\sqrt{4\pi}\Phi(Vt) + 2k_F Vt\right) \right].$$

The term proportional to $\partial_x \Phi(Vt)$ slowly redefines the chemical potential, and its action can be absorbed into the Luttinger Hamiltonian (B.1) by a redefinition of the field $\Phi \rightarrow \partial_x \Phi - \frac{K}{v_c} \int_{Vt}^x dx' U(x')$.

The remaining details of the calculation of the first-order response of the backscattering operator $I_b = \frac{dN_R}{dt} - \frac{dN_L}{dt}$ to the time-dependent perturbation $\delta\mathcal{H}(t)$ are included in Appendix B.1. The method is hereby summarized.

One writes the perturbation $\delta\mathcal{H}$ as a linear combination $\delta\mathcal{H}(t) = \sum_{\pm} \mathcal{C}_{\pm}^{\mathcal{H}} V_{\pm}$ where $V_{\pm} = e^{\pm\sqrt{4\pi}\Phi(Vt)}$ are vertex operators studied in the free bosonic theory (Appendix A.2). The backscattering current can also be shown to be linear combinations of these operators: $I_b = \sum_{\pm} \mathcal{C}_{\pm}^I V_{\pm}$.

The theory of out of equilibrium response gives the following expression regarding the response of I_b due to perturbation $\delta\mathcal{H}$ to first non-vanishing order in U_0 (Kubo formula):

$$\langle I_b \rangle_{\mathcal{H}+\delta\mathcal{H}} \approx \underbrace{\langle I_b \rangle_{\mathcal{H}}}_0 + i \int_{-\infty}^0 dt' \langle [I_b(0), \delta\mathcal{H}(t')] \rangle_{\mathcal{H}} . \quad (1.15)$$

This integral may be evaluated as an integral on a *Keldysh contour* [22] in the complex plane, granting a correct time-ordering of the two-point correlation functions. Its calculation makes use of the correlation function of two vertex operators in the Luttinger liquid

$$\langle \exp i\beta\Phi(z, \bar{z}) \exp i\beta'\Phi(\omega, \bar{\omega}) \rangle_{\mathcal{H}} = \delta_{\beta, -\beta'} \left(\frac{\alpha}{|z - \omega|} \right)^{K\beta^2/2\pi} .$$

After some simplifications, we obtain:

$$\langle I_b \rangle_{\mathcal{H}+\delta\mathcal{H}} = \frac{U_0^2}{\pi^2 \Gamma(2K)} \frac{(2\alpha k_F |V|)^{2(K-1)}}{(v_c^2 - V^2)^K} 2k_F V. \quad (1.16)$$

with Γ being the Euler Gamma function. In order to still have the right to apply Bosonization in our perturbed system (weak coupling), the typical energy exchanged in the $2k_F$ processes, $2k_F V$, must be small compared to the bandwidth $\sim k_F v_F$. Therefore the expression is valid only in the regime $V \ll v$ and the denominator of (1.16) may be simplified accordingly:

$$\langle I_b \rangle_{\mathcal{H}+\delta\mathcal{H}} = \frac{\text{sign}(V)}{\pi^2 \Gamma(2K)} \frac{U_0^2}{\alpha v} \left(2\alpha k_F \frac{|V|}{v} \right)^{2K-1}. \quad (1.17)$$

Equation (1.17) was calculated in an infinite system; in a finite system it is not valid under a velocity $V_{\text{low}} = v/N$ below which the energy exchanged in a $2k_F$ process is less than the energy of the first bosonic excitation, hence no excitations are created in the system by the motion of the potential.

We see that I_b depends on V as a power law with an exponent depending of the interaction strength $2K - 1$: this shows that quantum stirring can be used as a probe of the interactions, as was expected. In the absence of interactions, $K = 1$ and I_b becomes linear in V . Repulsive interactions $K < 1$ make stirring more efficient. As the Luttinger parameter is proportional [8] to the compressibility $\kappa = \partial\rho/\partial\mu = K/(v\pi^2)$, one sees that the result is in accordance with the intuition that “the smaller the compressibility, the larger the efficiency of stirring”.

This behavior agrees with the paper by Citro *et al.* [20] on quantum stirring of interacting bosons in one dimension. This comes as no surprise since both interacting bosons and interacting spinless fermions are described by the same effective Luttinger Liquid theory at low energy.

2 Fermions with spin $s = 1/2$

We now investigate the stirring of interacting fermions with two internal degrees of freedom $\sigma = \uparrow, \downarrow$. As discussed in the introduction, these degrees of freedom may describe two hyperfine spin states, two electronic orbital configurations or two species of atoms in a mixture.

At low temperature, the kinetic energy term of both fermionic components can be described in the tight-binding approximation. The new degree of freedom is thus added to the free Hamiltonian (1.1):

$$\mathcal{H}_0 = - \sum_{i,\sigma} t_\sigma (c_{i,\sigma}^\dagger c_{i+1,\sigma} + c_{i+1,\sigma}^\dagger c_{i,\sigma}).$$

We are interested in the case where both components have symmetrical roles, and the model exhibits an $SU(2)$ symmetry. This is the case for instance for a hyperfine multiplet $F = 1/2$ trapped in an all-optical lattice (no magnetic field). From now on both hopping amplitudes will therefore be equal $t_\uparrow = t_\downarrow = t$.

In the spinless case, we had to consider less natural nearest-neighbours interactions because two fermions could not be on the same site. The spin degeneracy now allows for two fermions to be on the same site, giving rise naturally to an on-site interaction energy. One thus obtains the so-called Hubbard Hamiltonian:

$$\mathcal{H} = -t \sum_{i,\sigma} (c_{i,\sigma}^\dagger c_{i+1,\sigma} + \text{h.c.}) + U \sum_i n_{i\uparrow} n_{i\downarrow}. \quad (2.1)$$

There is a competition between the kinetic term \mathcal{H}_0 and the potential energy $\mathcal{V} = U \sum_i n_{i\uparrow} n_{i\downarrow}$. The interaction term quantifies the gain or cost in energy of putting two fermions on the same site, mimicking for example the screening of Coulomb interactions. For instance, if t and U are both positive, the kinetic term tends to delocalize the fermions, while the cost in energy $U > 0$ of putting two fermions on the same site tends to minimize the overlaps (the interactions are repulsive). The competition between these antagonist effects gives rise to a transition between a metallic and a Mott-insulating phase. Moreover, as we will see, the ‘‘collectivization’’ of elementary excitations gives rise to the phenomenon of spin-charge separation: there are two elementary excitations, one being density fluctuations (charge sector) and the other being spin waves (spin sector). One will see that the supplementary degree of freedom with respect to the previous model greatly enriches the behaviors of the system.

Despite its apparent simplicity, the Hubbard model has strong experimental relevance; Hamiltonian (2.1) can be derived as a low-energy limit of the fermionic scattering processes at stake in a fermionic gas trapped on a lattice, as done *e.g.* in thesis [23].

Hamiltonian (2.1) has a $U(2) = U(1) \times SU(2)$ symmetry associated with the transformations $c_{i\alpha} \rightarrow e^{i\theta} c_{i\alpha}$ ($\theta \in \mathbb{R}$), $c_{i\alpha} \rightarrow \sum_\beta U_{\alpha\beta} c_{i\beta}$ ($U \in SU(2)$). These symmetries express the conservation of the charge and the spin invariance under a $SU(2)$ rotation. The Hamiltonian can be written in terms of the generators of these symmetries $\mathcal{N} = \sum_{i\alpha} n_{i\alpha}$ and $\vec{S} = \sum_i \vec{S}_i$ where $\vec{S}_i = \frac{1}{2} \sum_{\alpha,\beta} c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta}$ and $\vec{\sigma}$ are the Pauli matrices:

$$\mathcal{H} = -t \sum_{i,\sigma} (c_{i,\sigma}^\dagger c_{i+1,\sigma} + \text{h.c.}) - \frac{2U}{3} \sum_i S_i^2 + \mathcal{N} \frac{U}{2}$$

which exhibits the $SU(2)$ symmetry more clearly. At exactly half-filling, i.e. when $k_F = \pi/2\alpha$, a larger symmetry group emerges, as $U(1)$ in the charge sector is enlarged to $SU(2)$: in that case, the larger symmetry group for the whole Hamiltonian is $SO(4)$.

2.1 Hamiltonian and Bosonization

We shall now study the Hubbard model in the continuum limit. As before we place ourselves in the weak coupling limit $|U| \ll t$ in order to be able to use bosonization safely.

The continuum limit of the kinetic term is identical to what was obtained in the case of fermions with no degrees of freedom 1.1, but the fermions now carry a spin index σ . One obtains

$$\mathcal{H}_0 = -iv_F \sum_{\sigma=\uparrow,\downarrow} \int dx \left[: R_\sigma^\dagger(x) \partial_x R_\sigma(x) : - : L_\sigma^\dagger(x) \partial_x L_\sigma(x) : \right]. \quad (2.2)$$

The continuum limit of the annihilation operator on the lattice is given by

$$c_{j\sigma} \sim \sqrt{\alpha} \left[R_\sigma(x) e^{ik_F x} + L_\sigma(x) e^{-ik_F x} \right] \Big|_{x=j\alpha}. \quad (2.3)$$

This expression makes it possible to express the continuum limit of the potential energy in the Hamiltonian (2.1). Using this approach, the Hamiltonian decomposes as the sum of four terms

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} + \mathcal{H}_{\text{int}}^{2k_F} + \mathcal{H}_{\text{int}}^{4k_F} \quad (2.4)$$

where \mathcal{H}^{2k_F} and \mathcal{H}^{4k_F} are integrals of terms proportional to $e^{\pm 2ik_F x}$ and $e^{\pm 4ik_F x}$. As these terms oscillate quickly, we may safely neglect them as we did in equation (1.10), except at half-filling, where $\mathcal{H}_{\text{int}}^{4k_F}$ doesn't oscillate anymore. We will see that the physics in the charge sector are quite different at half-filling due to this supplementary term, called an *Umklapp* term.

The bosonization procedure can be adapted in order to handle the new degrees of freedom. Equation (1.6) becomes:

$$R_\sigma = \frac{\kappa_\sigma}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\phi_{R\sigma}}, \quad L_\sigma = \frac{\kappa_\sigma}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\phi_{L\sigma}} \quad (2.5)$$

where the supplementary factors κ_σ are Klein factors and ensure anticommutation between fermions of distinct spin by fulfilling the anticommutation relation $\{\kappa_\sigma, \kappa_{\sigma'}\} = 2\delta_{\sigma\sigma'}$.

The interaction term in (2.4) is expressed in the basis $\Phi_\sigma = \phi_{L\sigma} + \phi_{R\sigma}$, $\Theta_\sigma = \phi_{L\sigma} - \phi_{R\sigma}$ as

$$\mathcal{H}_{\text{int}} = \frac{\alpha V}{\pi} \int dx \partial_x \Phi_\uparrow \partial_x \Phi_\downarrow + \frac{V}{2\pi^2\alpha} \int dx \cos \left[\sqrt{4\pi}(\Phi_\uparrow - \Phi_\downarrow) \right]$$

while the Umklapp term expresses as

$$\mathcal{H}_U = \mathcal{H}_{\text{int}}^{4k_F} = -\frac{V}{2\pi^2\alpha} \int dx \cos \left[\sqrt{4\pi}(\Phi_\uparrow + \Phi_\downarrow) + 4k_F x \right].$$

The next step is to change basis in order to decouple spin and charge degrees of freedom. By a canonical transformation, we introduce the fields in the charge sector $\phi_{\alpha C} = \frac{1}{\sqrt{2}}(\phi_{\alpha\uparrow} + \phi_{\alpha\downarrow})$ and in the spin sector $\phi_{\alpha S} = \frac{1}{\sqrt{2}}(\phi_{\alpha\uparrow} - \phi_{\alpha\downarrow})$, where $\alpha = L, R$. In this basis, the Hubbard Hamiltonian splits into two commuting part : $\mathcal{H} = \mathcal{H}_C + \mathcal{H}_S$, where

$$\mathcal{H}_C = \frac{v_F}{2} \int dx \left[(\partial_x \Phi_C)^2 + (\partial_x \Theta_C)^2 \right] + \frac{\alpha U}{2\pi} \int dx (\partial_x \Phi_C)^2 \quad [+ \mathcal{H}_U \text{ at half-filling}] \quad (2.6)$$

$$\mathcal{H}_S = \frac{v_F}{2} \int dx \left[(\partial_x \Phi_S)^2 + (\partial_x \Theta_S)^2 \right] - \frac{\alpha U}{2\pi} \int dx (\partial_x \Phi_S)^2 + \frac{U}{2\pi^2\alpha} \int dx \cos \left(\sqrt{8\pi} \Phi_S \right) \quad (2.7)$$

and $\mathcal{H}_U = -\frac{V}{2\pi^2\alpha} \int dx \cos(\sqrt{8\pi}\Phi_C)$. This is the spin-charge separation: at low energy, fermionic excitations split into two decoupled excitations: charge fluctuations, or *holons*, and spin waves, or *spinons*.

2.2 Equilibrium properties of the Hubbard model

At incommensurable filling

If we are not at half-filling, the Hamiltonian in the charge sector (2.6) takes the form of the Luttinger Hamiltonian discussed earlier. It can be put in the canonical form (1.12) with a charge velocity $v_C = v_F \sqrt{1 + \frac{\alpha U}{\pi v_F}} = v_F + \frac{\alpha U}{2\pi} + \dots$ and a Luttinger parameter $K_C = \left(1 + \frac{\alpha U}{\pi v_F}\right)^{-1/2} = 1 - \frac{\alpha U}{2\pi v_F} + \dots$. We have as before $K_C > 1$ if the interactions are repulsive and $K_C < 1$ if they are attractive. The exact solution of this model away from weak coupling would show [24] that K_C has a lower bound $1/2$.

We conclude that the model is critical in the charge sector, and described by massless bosonic excitations.

The Hamiltonian in the spin sector (2.7) is more difficult to apprehend, because of the presence of the non-linear cosine term. We may use the renormalization group approach to understand whether this term has an impact on the low-energy physics of our system. Its scaling dimension being equal to the dimension of space-time $d = 2$, this term is *marginal* in the renormalization group sense, which means we cannot tell from the scaling dimension if it is *relevant* or *irrelevant* in our low-energy limit, and one has to push the analysis of the renormalization group flow further in order to elucidate that question [21]. The result is as follow: if U is positive, the spin sector “flows” to a critical model where $K_S = 1$, hence to a free theory. If U is negative, the cosine term is relevant and may not be dropped out. Fortunately, Hamiltonian (2.7) takes the form of the Sine-Gordon Hamiltonian, one of the few exactly solvable models, and it can be proved that in this case the spin sector is massive, which means there is a gap in energy between the fundamental and the first excited states, and at low-energy the field in the spin sector Φ_S is freezed in a minimum of the potential. The value taken by this field may be determined in a semi-classical way; imposing $\langle \cos(\sqrt{8\pi}\Phi_S) \rangle = 1$, one sees that $\langle \Phi_S \rangle = 0$, and the dual field Θ_S is completely delocalized. In this massive phase, the spin degrees of freedom display dimerization, with order parameter $\sum_i (-1)^i \langle \vec{S}_i \vec{S}_{i+1} \rangle \neq 0$.

One may wonder the reason for the value $K_S = 1$ for the Luttinger parameter in the spin sector. As might be seen from the computation of the correlation functions of the spin waves, this value stems from the spin $SU(2)$ symmetry, and ensures the invariance of these correlation functions under $SU(2)$ rotations.

At half-filling

At half-filling, the Umklapp term \mathcal{H}_U adds to the charge Hamiltonian (2.6). The charge Hamiltonian also becomes a Sine-Gordon model, with opposite signs in front of U with respect to the spin sector. The properties of the spin sector are untouched, and the properties of the charge sector are deduced from the earlier discussion on the Sine-Gordon model. Hence the charge sector is critical with $K_C = 1$ if $U < 0$, and massive if $U > 0$. The gaped phase is a Mott insulator, as its insulating character stems only from the interactions. The emergence of the $SU(2)$ particle-hole symmetry in the charge sector at half filling imposes the value $K_C = 1$ in the case of attractive interactions.

Summary

The following table summarizes the equilibrium properties of the Hubbard model.

| | | $U < 0$ | $U > 0$ |
|--------|------------------------|---------------------|---------------------------|
| Charge | incommensurate filling | critical, $K_C > 1$ | critical, $1/2 < K_C < 1$ |
| | half-filling | critical, $K_C = 1$ | gaped, Mott insulator |
| Spin | | gapped, dimerized | critical, $K_S = 1$ |

2.3 Response to stirring

We will now investigate the response of the two-component fermionic gas to quantum stirring. The formalism developed in Subsection 1.3 has to be adapted in order to account for the added degrees of freedom.

The expression of the fermionic density $n(x) = c^\dagger(x)c(x)$ to which the potential $U(x - Vt)$ couples can be determined using the expression of the fermionic annihilation operator in the continuum limit (2.3). It can be shown to contain contributions both from the charge and from the spin sector:

$$n(x) = \sqrt{\frac{2}{\pi}} \partial_x \Phi_C - \frac{2}{\pi\alpha} \sin\left(\sqrt{2\pi}\Phi_C + 2k_F x\right) \cos(\sqrt{2\pi}\Phi_S). \quad (2.8)$$

The Hamiltonian describing the perturbation reads as before $\delta\mathcal{H} = U_0 n(x = Vt)$. When the charge sector is critical, the term proportional to $\partial_x \Phi_C$ in $\delta\mathcal{H}$ may as before be absorbed in the Luttinger Hamiltonian of the charge sector. We will suppose this is the case, hence the following study will not cover half-filling with attractive interactions where, according to the last subsection, the charge sector is in a Mott insulating phase.

We may now define two separate currents backscattering currents, one for each degree of freedom I_b^\uparrow and I_b^\downarrow . However, since the coupling of the perturbation is the same to \uparrow -fermions as to \downarrow -fermions, the perturbation does not break the $SU(2)$ symmetry of the model, hence $I_b^\uparrow = I_b^\downarrow$. In the following we define the total backscattering current $I_b = I_b^\uparrow + I_b^\downarrow$.

The remaining details of the calculation of the response of I_b^\uparrow to the time-dependent perturbation $\delta\mathcal{H}(t)$ will now be summarized; more details on the calculation are given in Appendix B.2.

One expresses the perturbation $\delta\mathcal{H}$ and the operator I_b^\uparrow as linear combinations of vertex operators in the charge sector $V_\pm^C = e^{\pm i\sqrt{2\pi}\Phi_C(Vt)}$ and in the spin sector $V_\pm^S = e^{\pm i\sqrt{2\pi}\Phi_S(Vt)}$:

$$\delta\mathcal{H}(t) = \sum_{\epsilon, \mu \in \{-1, 1\}} c_\epsilon^H V_\epsilon^C V_\mu^S, \quad I_b^\uparrow = \sum_{\pm} C_\pm^I V_\pm^C V_\mp^S.$$

Using the same Keldysh formalism as in Subsection 1.3, we obtain the following response at first non-vanishing order in U_0 .

$$\langle I_b^\uparrow \rangle_{\mathcal{H} + \delta\mathcal{H}} \approx iT_C \int_C dt' \sum_{\pm} C_\pm^I(0) C_\mp^H(t') \langle V_\pm^C(0) V_\mp^C(t') \rangle_{\mathcal{H}_C} \langle V_\mp^S(0) V_\pm^S(t') \rangle_{\mathcal{H}_S}. \quad (2.9)$$

Because of spin-charge separation, correlators of the vertex operators in the charge and in the spin sector appear separately in expression (2.9). We restricted ourselves to the cases where the charge sector is critical; we now have to distinguish whether the spin sector is critical (if $U > 0$) or dimerized (if $U < 0$).

Critical spin sector, $U > 0$

If the fermionic interactions are repulsive, both sectors are critical, $1/2 < K_C < 1$ and $K_S = 1$, hence

$$\langle V_{\pm}^C(0)V_{\mp}^C(t') \rangle_{\mathcal{H}_C} = \left(\frac{\alpha}{|z - \omega|} \right)^{K_C}, \quad \langle V_{\mp}^S(0)V_{\pm}^S(t') \rangle_{\mathcal{H}_S} = \left(\frac{\alpha}{|z - \omega|} \right)$$

and the calculation (2.9) amounts to replacing $2K$ by $K_S + K_C = 1 + K_C$ with respects to the one-component case (1.17):

$$\langle I_b \rangle_{\mathcal{H} + \delta\mathcal{H}} = \frac{\text{sign}(V)}{\pi^2 \Gamma(1 + K_C)} \frac{U_0^2}{\alpha v} \left(2\alpha k_F \frac{|V|}{v} \right)^{K_C}. \quad (2.10)$$

Dimerized spin sector, $U < 0$

If the fermionic interactions are attractive, the charge sector is critical: $K_C \geq 1$, with $K_C = 1$ at half-filling. The spin sector is massive. In this regime, the term $\cos \sqrt{8\pi}\Phi_S$ in the spin Hamiltonian is relevant and the field Φ_S is frozen in the minimum of the potential. Vertex operators in the spin sector have a non-zero mean value, which changes the structure of the correlations appearing in (2.9). We therefore expect the efficiency of quantum stirring to be suddenly enhanced when a spin gap opens.

The correlation functions of the vertex operators in the spin sector have the form $\langle V^\dagger(x, 0)V(0, 0) \rangle = |\langle V(0, 0) \rangle|^2 + \lambda \exp(-x/\xi)$ with λ a constant, $\xi = v_S/\Delta_S$ the correlation length and Δ_S the gap. The mean value $|\langle V_{\pm}^S(0) \rangle_{\mathcal{H}_S}|$ can be determined exactly in the sine-Gordon model [25]. Restricting ourselves to a dimensional analysis, $|\langle V_{\pm}^S(0) \rangle_{\mathcal{H}_S}| = \gamma \Delta_S^{\beta_S^2/4\pi} = \gamma \Delta_S^{1/2}$ with $\beta_S = \sqrt{2\pi}K_S = \sqrt{2\pi}$, where γ is a constant. At low energy $\langle V^\dagger(x, 0)V(0, 0) \rangle = \gamma^2 \Delta_S$ can be taken out of the integral (2.9).

The correlators in the charge sector are still

$$\langle V_{\pm}^C(0)V_{\mp}^C(t') \rangle_{\mathcal{H}_C} = \left(\frac{\alpha}{|z - \omega|} \right)^{K_C}.$$

Hence we may again use the result of the spinless version (1.17) by replacing $2K$ by K_C (and multiplying the result by $\gamma^2 \Delta_S$). The final result is thus:

$$\langle I_b \rangle_{\mathcal{H}_0 + \delta\mathcal{H}} = \frac{\text{sign}(V)}{\pi^2 \Gamma(K_C)} \frac{U_0^2 \gamma^2 \Delta_S}{\alpha v} \left(2\alpha k_F \frac{|V|}{v} \right)^{K_C - 1}. \quad (2.11)$$

Discussion

The expressions of $\langle I_b \rangle$ obtained in the case of repulsive and attractive interactions are both power laws in V with an exponent depending on the strength of the interactions. The main difference between the different phases is the value of this exponent. A sketch of the evolution of the exponent is presented on Figure 2.1. Again, more repulsive interactions of translate in lower compressibility and higher stirring efficiency. The apparition of a gaped phase boosts the efficiency of the stirring, as we can see from the ‘‘efficiency jump’’ in the exponent around $U = 0$. This efficiency jump allows, in principle, to locate the phase transition between the critical and the dimerized spin sector in the gas of fermions with two components.

This result must be interpreted with caution. The stirring in the dimerized spin phase has the form (2.11) only if the energy scales involved in the stirring are small compared

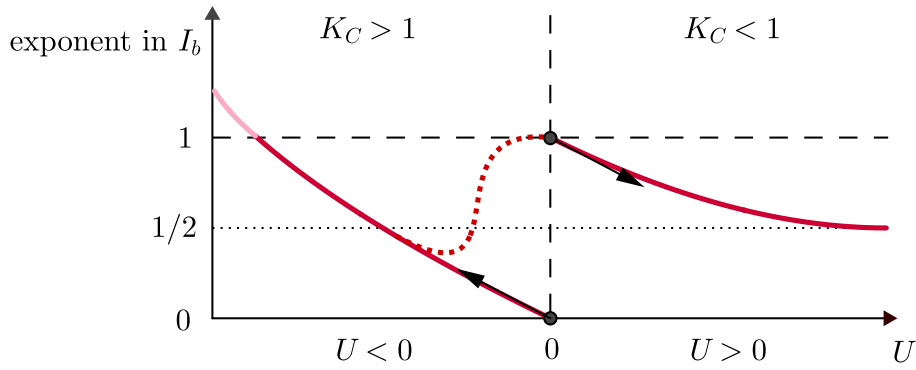


Figure 2.1 – Sketch of the variation of the exponent of V in I_b as a function of the interaction strength U . The dotted line suggests the effect of gap opening, which smoothes the transition.

to the gap energy Δ_S , i.e. if the gap is “robust” against stirring and the spin degrees of freedom stay frozen. This puts a limit to the stirring velocity $2k_F V \ll \Delta_S$. In the opposite limit $2k_F V \gg \Delta_S$, the correlators in the spin sector in (2.9) are calculated on distances which are too small to be sensitive to the gap, and expression (2.11) is replaced by the critical result (2.10). The limit $2k_F V \ll \Delta_S$ becomes restrictive when $U \rightarrow 0^-$ where the gap $\Delta_S \sim \exp(-1/|U|)$ is very small. The discontinuity is therefore smoothed in the region $U \lesssim 0$, as suggested by the dotted line on Figure 2.1.

Conclusion

This internship covered the theoretical study of quantum stirring as a probe of unidimensional fermionic gases with one and two internal degrees of freedom. The analysis made use of bosonization to map the fermions to a bosonic system, and treated the interaction between the laser and the fluid perturbatively in order to identify the scaling law of the backscattering current I_b with respect to the stirring velocity.

In the one-component Fermi gas, the backscattering current measured at short stirring times depend on the stirring velocity as a power law. The exponent $2K - 1$ depends linearly on the Luttinger parameter K . A measure of I_b allows thus a characterization of the Luttinger parameter, characterizing the interaction strength and compressibility of the fluid. Repulsive interactions make stirring more efficient.

A two component fermionic gas exhibits spin-charge separation, and both the spin and the charge sector can either be critical or massive, depending on the sign of the interactions. The analysis shows that repulsive interactions still favour stirring, while the opening of a gap in the spin sector translates in a non-monotony of the scaling law exponent. The gap in the spin sector translates in a “boost” in stirring efficiency.

Quantum stirring is an elegant way of probing the collective excitations of an ultracold system. In the light of these promising results, the study could be pushed forward in several ways. The Mott insulating phase is more difficult to analyse, as it does not fit well in the picture of second-order perturbations. A more detailed perturbation analysis could allow a better control of the effect of gap opening, and the use of a renormalization group approach could complement the study in some of its limits.

As an extension of this study, one could consider the effects of $SU(2)$ symmetry breaking by using a laser mode addressing each spin degree of freedom with different couplings U_\uparrow, U_\downarrow . Similarly to the case of the Mott insulator, this requires the use of a more precise perturbation approach in the spin sector.

Finally, quantum stirring of systems with $N > 2$ internal components promises to be an interesting way of characterizing their exotic physics.

Remerciements

J’adresse un grand merci à Édouard Boulat pour sa disponibilité sans faille et ses explications patientes et lumineuses qui m’ont permis de me familiariser peu à peu avec les outils utilisés au cours de mon stage. Mes remerciements vont également à Philippe Lecheminant pour ses discussions amicales et avisées, et ses excellentes notes de cours sur la bosonisation. Enfin, merci aux responsables de l’ICFP pour l’organisation et le suivi des stages de M2.

A Bosonization

This appendix presents a quick introduction to the Bosonization approach, and summaries some of its important results. It is based on lecture notes written by P. Lecheminant.

Bosonization maps low energy excitations of an unidimensional system of free fermions to a system of free bosons. It stems from the $U(1)_R \times U(1)_L$ symmetry present both in the model of free fermions and in the model of free bosons in 1+1 dimensions.

Because of the collective behavior of the excitations in one dimension, Bosonization often provides a better basis to study a fermionic system than the original fermions.

Subsection A.1 and A.2 reminds important aspects of the fermionic and bosonic theories in one dimension. Subsection A.3 exhibits the correspondance between the two models.

A.1 Free fermions in 1+1 dimensions

As seen in Subsection 1.1, the Hamiltonian describing free fermions on a lattice is given by:

$$\mathcal{H}_0 = -t \sum_{n=1}^N c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n$$

and is expressed in the continuum limit in terms of the fields in the right and left sector as

$$\mathcal{H}_0 = -iv_F \int_0^L dx \left[:R^\dagger(x)\partial_x R(x): - :L^\dagger(x)\partial_x L(x): \right].$$

This is the massless Dirac lagrangian in 1+1 dimensions and exhibits a $U(1)_R \times U(1)_L$ symmetry (conformal invariance) associated to the chiral currents $J_R = :R^\dagger R:$ and $J_L = :L^\dagger L:$.

In a Conformal Field Theory, every operator A has conformal weights (h, \bar{h}) which fixes its two-point correlation function up to a multiplicative constant by:

$$\langle A(z, \bar{z}) A^\dagger(\omega, \bar{\omega}) \rangle \sim \frac{1}{(z - \omega)^{2h} (\bar{z} - \bar{\omega})^{2\bar{h}}} \quad (\text{A.1})$$

where $z = v_F \tau + ix$ and $\omega = v_F \tau' + ix$ are the spacetime coordinates after a Wick imaginary time rotation $\tau = it$. The divergence of the Green functions (A.1) when $z \rightarrow \omega$ is a direct evidence that our model fails to account for the physics of the system at high energy.

The quantity $s = h - \bar{h}$ is the conformal spin of the operator, whereas $\Delta = h + \bar{h}$ is its scaling dimension. These characteristic numbers attached to the fields are essential aspects of the theory. They may be determined using counting arguments, since the weights of products or derivatives of operators are easily deduced from the conformal weights of these operators.

The free fermions correlators express as:

$$\langle L^\dagger(z) L(\omega) \rangle = \frac{1}{2\pi(z - \omega)} \quad \langle R^\dagger(\bar{z}) R(\bar{\omega}) \rangle = \frac{1}{2\pi(\bar{z} - \bar{\omega})} \quad (\text{A.2})$$

which shows that L has conformal weights $(1/2, 0)$ and R has conformal weights $(0, 1/2)$.

The UV cutoff may be introduced in these correlators by replacing $v_F \tau$ by $v_F \tau + \alpha$ in z . This preserves time-ordering and is of importance when calculating e.g. the difference between two correlators, as in the calculation of the response to stirring in Appendix B.

Within the framework of CFT, the product of two operators, which is divergent when taken at the same point because of the UV singularity, can be expanded as a sum over operators in the form of a Laurent series:

$$A(z) B(\omega) \sim \sum_{n=1}^{N_0} \frac{c_n(\omega)}{(z - \omega)^n} + :AB: (\omega) + O(z - \omega).$$

This operator product expansion (OPE) will have a meaning when inserted into a correlation function (imposing $\Re z > \Re \omega$ in order to conserve time-ordering). The OPE of the fermionic operators may be derived using mode decomposition and a Taylor expansion inside the normally ordered term. One obtains:

$$\begin{aligned} L^\dagger(z)L(\omega) &\sim \frac{1}{2\pi(z-\omega)} + J_L(\omega) + \mathcal{O}(z-\omega) \\ R^\dagger(\bar{z})R(\bar{\omega}) &\sim \frac{1}{2\pi(\bar{z}-\bar{\omega})} + J_R(\bar{\omega}) + \mathcal{O}(\bar{z}-\bar{\omega}). \end{aligned} \quad (\text{A.3})$$

To compute the OPEs of composite operators, in the case of a Gaussian Hamiltonian one may use Wick contractions. This allows one to compute the OPEs of the currents:

$$\begin{aligned} J_L(z)J_L(\omega) &\sim \frac{1}{4\pi^2(z-\omega)^2} - \frac{1}{\pi} :L^\dagger\partial L: (\omega) + \mathcal{O}(z-\omega) \\ J_R(\bar{z})J_R(\bar{\omega}) &\sim \frac{1}{4\pi^2(\bar{z}-\bar{\omega})^2} - \frac{1}{\pi} :R^\dagger\bar{\partial}R: (\bar{\omega}) + \mathcal{O}(\bar{z}-\bar{\omega}) \end{aligned} \quad (\text{A.4})$$

where $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$. The two-point correlation functions are given directly by the leftmost totally contracted term in the OPEs, hence J_L has conformal weights $(1, 0)$ and J_R has conformal weights $(0, 1)$, which could be deduced from a counting argument.

The equal-time commutation relations between J_L and J_R are called *anomalous*. As explained in [26], they must be carefully calculated from the OPEs (A.4) in a way which accounts for the time-ordering of the Green functions:

$$[A(x), B(y)]^\mp = \lim_{\tau \rightarrow 0} [A(\tau, x)B(0, y) - \epsilon A(-\tau, x)B(0, y)] \quad (\text{A.5})$$

where $\epsilon = -1$ in the presence of fermionic fields, and $\epsilon = +1$ in the presence of bosonic fields. One obtains the commutation relations:

$$[J_R(x), J_R(y)] = -\frac{i}{2\pi}\delta'(x-y), \quad [J_L(x), J_L(y)] = \frac{i}{2\pi}\delta'(x-y).$$

and $[J_L(x), J_R(y)] = 0$. The expression (1.5) of the Hamiltonian \mathcal{H}_0 in terms of the chiral currents is deduced from the OPEs (A.4):

$$\mathcal{H}_0 = \pi v_F \int_0^L dx \left[:J_L^2: + :J_R^2: \right]. \quad (\text{A.6})$$

A.2 Free bosons in 1+1 dimensions

The Lagrangian of a one-dimensional free bosonic field reads:

$$\mathcal{L}_0 = \frac{1}{2} \int dx \left[\frac{1}{v} (\partial_t \Phi)^2 - v (\partial_x \Phi)^2 \right].$$

The associated Hamiltonian expresses in terms of the conjugate momentum $\Pi = \frac{1}{v} \partial_t \Phi$:

$$\mathcal{H}_0 = \frac{v}{2} \int dx \left[:\Pi^2: + :(\partial_x \Phi)^2: \right].$$

The equation of motion associated to Lagrangian \mathcal{L}_0 is Klein-Gordon equation:

$$\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \Phi = 0 \quad (\text{A.7})$$

hence Φ decomposes as the sum of two independent chiral fields $\Phi = \phi_L(vt + x) + \phi_R(vt - x)$. Equation (A.7) is equivalent to imposing ϕ_L to be holomorphic in $z = v\tau - ix$ and ϕ_R anti-holomorphic in z , hence $\phi_L(z, \bar{z}) = \phi_L(z)$ and $\phi_R(z, \bar{z}) = \phi_R(\bar{z})$. From this one can easily show that Hamiltonian \mathcal{H}_0 has a simple expression in terms of the total field $\Phi = \phi_L + \phi_R$ and its dual field $\Theta = \phi_L - \phi_R$:

$$\mathcal{H}_0 = v \int dx \left[:(\partial_x \Phi)^2: + :(\partial_x \Theta)^2: \right].$$

This exhibits the $U(1)_R \times U(1)_L$ symmetry of the model, associated to the transformations $\phi_L, \phi_R \rightarrow \phi_L + \alpha_L, \phi_R + \alpha_R$. The corresponding conserved chiral currents express as

$$J_L = \frac{1}{\sqrt{\pi}} : \partial_x \phi_L : \quad J_R = \frac{1}{\sqrt{\pi}} : \partial_x \phi_R :$$

hence \mathcal{H}_0 may be written in terms of these currents

$$\mathcal{H}_0 = \pi v \int dx \left[:J_L^2: + :J_R^2: \right]. \quad (\text{A.8})$$

The commutation relations between the chiral fields may be determined using the canonical commutation relations satisfied by Π and ϕ : $[\phi(x), \phi(y)] = [\Pi(x), \Pi(y)] = 0$ and $[\phi(x), \Pi(y)] = i\delta(x - y)$. One obtains $[\phi^R(x), \phi^R(x')] = \frac{i}{4} \text{sgn}(x - x')$, $[\phi^L(x), \phi^L(x')] = -\frac{i}{4} \text{sgn}(x - x')$ and $[\phi^R(x), \phi^L(x')] = \frac{i}{4}$.

By derivation one obtains the fact J_L and J_R commute, and within each sector $[J_L(x), J_L(y)] = \frac{i}{2\pi} \delta'(x - y)$ and $[J_R(x), J_R(y)] = -\frac{i}{2\pi} \delta'(x - y)$. The two chiral parts of Hamiltonian (A.8) commute.

The two-point correlation functions between the bosonic fields may be computed using mode decomposition of Φ and Θ in momentum space. One obtains:

$$\begin{aligned} \langle \phi_L(z) \phi_L(\omega) \rangle &= -\frac{1}{4\pi} \ln \left(\frac{z - \omega}{L} \right) \\ \langle \phi_R(z) \phi_R(\omega) \rangle &= -\frac{1}{4\pi} \ln \left(\frac{\bar{z} - \bar{\omega}}{L} \right) \\ \langle \phi_L(z) \phi_L(\omega) \rangle &= -\frac{1}{4\pi} \ln \left(\frac{|z - \omega|}{L} \right) \end{aligned}$$

hence

$$\begin{aligned} \langle J_L(z) J_L(\omega) \rangle &= \frac{1}{4\pi^2} \frac{1}{(z - \omega)^2} \\ \langle J_R(z) J_R(\omega) \rangle &= \frac{1}{4\pi} \frac{1}{(\bar{z} - \bar{\omega})^2}. \end{aligned}$$

Vertex operators

An important class of operators are introduced in the bosonic theory: called the vertex operators, these operators $e^{i\beta\phi_L}$, $e^{i\beta\phi_R}$, $e^{i\beta\Phi}$ with β a real number are obtained by exponentiation of the fields.

Using Baker-Campbell-Hausdorff identities

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}, \quad e^A e^B = e^B e^A e^{[A,B]}$$

valid when $[A, B]$ is a number, one may show the following two-point correlation functions for the vertex operators

$$\begin{aligned}
\langle \exp i\beta\phi_L(z) \exp i\beta'\phi_L(\omega) \rangle_{\mathcal{H}} &= \delta_{\beta, -\beta'} \left(\frac{\alpha}{z - \omega} \right)^{\beta^2/4\pi}, \\
\langle \exp i\beta\phi_R(\bar{z}) \exp i\beta'\phi_R(\bar{\omega}) \rangle_{\mathcal{H}} &= \delta_{\beta, -\beta'} \left(\frac{\alpha}{\bar{z} - \bar{\omega}} \right)^{\beta^2/4\pi}, \\
\langle \exp i\beta\Phi(z, \bar{z}) \exp i\beta'\Phi(\omega, \bar{\omega}) \rangle_{\mathcal{H}} &= \delta_{\beta, -\beta'} \left(\frac{\alpha}{|z - \omega|} \right)^{\beta^2/2\pi}.
\end{aligned} \tag{A.9}$$

As one can see, the vertex operators provide a family of operators of arbitrary scaling dimension. This is very important to bosonization, as it allows to identify the correct representation of the fermionic operators in the bosonic basis.

A.3 Bosonization dictionary

It is remarkable that the Hamiltonians of both the free fermionic and bosonic theories admit the same expressions (A.6) and (A.8) in terms of their conserved chiral currents, and that these currents present the same correlations and commutation relations. This motivates the search for a mapping that identifies both theories:

$$:L^\dagger L: \leftrightarrow : \frac{1}{\sqrt{\pi}} \partial_x \phi_L :, \quad :R^\dagger R: \leftrightarrow : \frac{1}{\sqrt{\pi}} \partial_x \phi_R :.$$

The search for operators of scaling dimension 1/2 to represent the fermionic operators in the bosonic theory suggests the use of vertex operators (A.9) with $\beta = \sqrt{4\pi}$. Choosing the right normalisation to equalize correlators (A.2) and (A.9), a natural choice is to set:

$$\begin{aligned}
L(z) &= \frac{1}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\phi_L(z)}, & R(\bar{z}) &= \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\phi_R(\bar{z})}, \\
L^\dagger(z) &= \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\phi_L(z)}, & R^\dagger(\bar{z}) &= \frac{1}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\phi_R(\bar{z})}.
\end{aligned}$$

Indeed, it can be shown that the correlation functions and the (anti-)commutation relations of the fields give the same results in the two theories using this identification. The equivalence of the two theories at the level of the operators is thus established.

The identification of the two pictures is summarized in the following table:

| Free Fermions | \longleftrightarrow | Free Bosons |
|--|-----------------------|---|
| $iv_F \bar{\Psi} \gamma^\mu \delta_\mu \Psi$ | \longleftrightarrow | $\frac{v_F}{2} \partial_\mu \Phi \partial^\mu \Phi$ |
| $iv_F (:L^\dagger \partial_x L: - :R^\dagger \partial_x R:)$ | \longleftrightarrow | $v_F [:(\partial_x \phi_R)^2: + :(\partial_x \phi_L)^2:]$ |
| $:R^\dagger R:$ | \longleftrightarrow | $:\frac{1}{\sqrt{\pi}} \partial_x \phi_R:$ |
| $:L^\dagger L:$ | \longleftrightarrow | $:\frac{1}{\sqrt{\pi}} \partial_x \phi_L:$ |
| L | \longleftrightarrow | $\frac{1}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\phi_L}$ |
| R | \longleftrightarrow | $\frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\phi_R}$ |

B Response to stirring

B.1 In the spinless model

We are interested in calculating the response of a Luttinger liquid described by Hamiltonian (1.12):

$$\mathcal{H} = \frac{v}{2} \int dx \frac{1}{K} :(\partial_x \Phi): + K :(\partial_x \Theta)^2: . \quad (\text{B.1})$$

The Hamiltonian of the perturbation reads $\delta\mathcal{H} = n(x = Vt)$ where in the continuum limit, the fermionic density expresses as equation (1.9), or in its bosonized version:

$$n(x) = \frac{1}{\sqrt{\pi}} \partial_x \Phi - \frac{1}{\pi\alpha} \sin\left(\sqrt{4\pi}\Phi + 2k_F x\right).$$

Therefore

$$\delta\mathcal{H}(t) = U_0 \left[\frac{1}{\sqrt{\pi}} \partial_x \Phi(Vt) - \frac{1}{\pi\alpha} \sin\left(\sqrt{4\pi}\Phi(Vt) + 2k_F Vt\right) \right].$$

The action of the term proportional to $\partial_x \Phi(Vt)$ can be absorbed into the Luttinger Hamiltonian (B.1) by a redefinition of the field $\phi \rightarrow \partial_x \phi - \frac{K}{v_c} \int_{Vt}^x dx' U(x')$.

Therefore, $\delta\mathcal{H}$ can be rewritten $\delta\mathcal{H}(t) = \sum_{\pm} \mathcal{C}_{\pm}^{\mathcal{H}} V_{\pm}$ where $V_{\pm} = e^{\pm\sqrt{4\pi}\Phi(Vt)}$ is a vertex operator in the free bosonic theory A.2 and $\mathcal{C}_{\pm}^{\mathcal{H}} = \mp \frac{U_0}{2i\pi\alpha} e^{\pm i\omega_b t}$, where we denote $\omega_b = 2k_F V$ the stirring frequency.

The backscattering current operator expresses as $I_b = \frac{dN_R}{dt} - \frac{dN_L}{dt} = 2\frac{dN_R}{dt}$ because the total Hamiltonian conserves the total charge $N_L + N_R$. The number of right or left movers express in terms of the chiral currents as $N_{R/L} = \int J_{L/R} = \frac{1}{\sqrt{\pi}} \int \partial_x \phi_{R/L}$, and $\frac{dN_R}{dt} = i[\mathcal{H}_{\text{tot}}, N_R] = i[\delta\mathcal{H}, N_R]$ because $\delta\mathcal{H}$ breaks the $U(1)_L \times U(1)_R$ symmetry. Hence

$$\frac{dN_R}{dt} = i \sum_{\pm} \int \mathcal{C}_{\pm}^{\mathcal{H}} [V_{\pm}, J_R(x)] dx$$

where the equal-time commutator can be deduced from the OPE $J_R(\bar{z})V_{\pm} = \frac{\mp 1}{2\pi(\bar{z}-\bar{w})} V_{\pm}$, which implies that $[J_R(x), V_{\pm}(Vt)] = \mp V_{\pm}(Vt)\delta(x - Vt)$, hence I_b can finally be expressed as $I_b = \sum_{\pm} \mathcal{C}_{\pm}^I V_{\pm}$ where $\mathcal{C}_{\pm}^I = -\frac{U_0}{\pi\alpha} e^{\pm i\omega_b t}$ and the stirring frequency $\omega_b = 2k_F V$.

Out of equilibrium response theory gives the following expression regarding the response of I_b due to perturbation $\delta\mathcal{H}$ to first non-vanishing order (Kubo formula):

$$\begin{aligned} \langle I_b \rangle_{\mathcal{H}_0 + \delta\mathcal{H}} &\approx \underbrace{\langle I_b \rangle_{\mathcal{H}}}_0 + i \int_{-\infty}^0 dt' \langle [I_b(0), \delta\mathcal{H}(t')] \rangle_{\mathcal{H}} \\ &= iT_{\mathcal{C}} \int_{\mathcal{C}} dt' \langle I_b \delta\mathcal{H}(z, \bar{z}) \rangle_{\mathcal{H}} \end{aligned} \quad (\text{B.2})$$

where the integral on the second line of (B.2) is calculated on a *Keldysh contour* [22] going from $-\infty$ to 0 with a infinitesimal negative imaginary part, and back from 0 to $-\infty$ with an infinitesimal positive imaginary part (this choice allows the correct time-ordering in the correlation functions). The time-ordering operator $T_{\mathcal{C}}$ exchanges the operators in the correlator on the “return” part of the contour.

$$\langle I \rangle_{\mathcal{H}_0 + \delta\mathcal{H}} = \frac{U_0^2}{2\pi^2\alpha^2} \sum_{\pm} \mp T_{\mathcal{C}} \int_{\mathcal{C}} dt' e^{\mp i\omega_b t'} \langle V_{\pm}(0) V_{\mp}(t') \rangle_{\mathcal{H}}$$

where we make use of the correlators of two vertex operators $\langle V_\beta(z, \bar{z}) V_{\beta'}(\omega, \bar{\omega}) \rangle_{\mathcal{H}} = \delta_{\beta, -\beta'} \left(\frac{\alpha}{|z-\omega|} \right)^{K\beta^2/2\pi}$.

$$\begin{aligned} \langle I \rangle_{\mathcal{H}_0 + \delta\mathcal{H}} &= \frac{U_0^2 \alpha^{2K-2}}{2\pi^2 (v_c^2 - V^2)^K} \sum_{\pm} \mp \left[\int_{-\infty}^0 dt' \frac{e^{\mp i\omega_b t'}}{|t'|^{2K} (e^{-i\pi^-})^K} + \int_0^{-\infty} dt' \frac{e^{\mp i\omega_b t'}}{|t'|^{2K} (e^{i\pi^-})^K} \right] \\ &= \frac{U_0^2 \alpha^{2K-2}}{2\pi^2 (v_c^2 - V^2)^K} 2i \sin(\pi K) \sum_{\pm} \mp \int_0^{\infty} d\tau \frac{e^{\pm i\omega_b \tau}}{\tau^{2K}} \end{aligned} \quad (\text{B.3})$$

and making use of complex analysis, $\int_0^{\infty} d\tau \frac{e^{i\omega_b \tau}}{\tau^{2K}} = \text{sgn}(\omega_b) i |\omega_b|^{2K-1} e^{-i \text{sgn}(\omega_b) \pi K} \Gamma(1-2K)$, hence after some simplifications and using $\omega_b = 2k_F V$:

$$\langle I \rangle_{\mathcal{H}_0 + \delta\mathcal{H}} = \frac{U_0^2}{\pi^2 \Gamma(2K)} \frac{(2\alpha k_F |V|)^{2(K-1)}}{(v_c^2 - V^2)^K} 2k_F V. \quad (\text{B.4})$$

B.2 In the Hubbard model

The Hamiltonian describing the perturbation expresses as

$$\delta\mathcal{H}(t) = -\frac{2U_0}{\pi\alpha} \sin\left(\sqrt{2\pi}\Phi_C(Vt) + 2k_F Vt\right) \cos(\sqrt{2\pi}\Phi_S) = \sum_{\epsilon, \mu \in \{-1, 1\}} \mathcal{C}_\epsilon^H V_\epsilon^C V_\mu^S$$

with $V_\epsilon^C = e^{i\epsilon\sqrt{2\pi}\Phi_C(Vt)}$ and $V_\mu^S = e^{i\mu\sqrt{2\pi}\Phi_S(Vt)}$, where $\mathcal{C}_\epsilon^H = -\epsilon \frac{U_0}{2i\pi\alpha} e^{i\epsilon\omega_b t}$ and $\omega_b = 2k_F V$.

The ‘‘up’’ current in the right sector expresses as $N_R^\uparrow = \frac{1}{\pi} \int \partial_x \phi_R^\uparrow$ and

$$\frac{dN_R^\uparrow}{dt} = i \sum_{\epsilon, \mu} \int C_\epsilon^H [V_\epsilon^C V_\mu^S, J_R^\uparrow(x)] dx$$

where $J_R^\uparrow(x) = (J_R^C(x) + J_R^S(x))/\sqrt{2}$, and using $[V_\epsilon^{C/S}, J_R^{C/S}(x)] = \epsilon \frac{1}{\sqrt{2}} V_\epsilon^{C/S} \delta(x - Vt)$, one obtains

$$I^\uparrow = 2 \frac{dN_R^\uparrow}{dt} = i \sum_{\epsilon, \mu} (\epsilon + \mu) C_\epsilon^H V_\epsilon^C V_\mu^S = i \sum_{\pm} \pm C_\pm^H V_\pm^C V_\mp^S = \sum_{\pm} C_\pm^I V_\pm^C V_\mp^S$$

where $C_\pm^I = -\frac{U_0}{2\pi\alpha} e^{\pm i\omega_b t}$. Because of $SU(2)$ spin conservation, $I^\uparrow = I^\downarrow$. As before in eq. (B.2), the perturbation can be evaluated using

$$\begin{aligned} \langle I^\uparrow \rangle_{\mathcal{H} + \delta\mathcal{H}} &\approx \underbrace{\langle I^\uparrow \rangle_{\mathcal{H}}}_0 + iT_C \int_C dt' \langle I^\uparrow \delta\mathcal{H}(z, \bar{z}) \rangle_{\mathcal{H}} \\ &= iT_C \int_C dt' \sum_{\pm} C_\pm^I(0) C_\mp^H(t') \langle V_\pm^C(0) V_\mp^C(t') \rangle_{\mathcal{H}_C} \langle V_\mp^S(0) V_\pm^S(t') \rangle_{\mathcal{H}_S} \end{aligned} \quad (\text{B.5})$$

where $z = i(v_C + V)t'$, $\bar{z} = i(v_C - V)t'$. This gives the result (2.9). The remaining calculations are discussed in Subsection 2.3 and are analogous to what has been done in the spinless case.

References

- [1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, “Observation of bose-einstein condensation in a dilute atomic vapor,” *science*, vol. 269, no. 5221, pp. 198–201, 1995.
- [2] S. N. Bose, “Plancks gesetz und lichtquantenhypothese,” *Z. phys*, vol. 26, no. 3, p. 178, 1924.
- [3] A. Einstein, *Quantentheorie des einatomigen idealen Gases*. Akademie der Wissenschaften, in Kommission bei W. de Gruyter, 1924.
- [4] B. DeMarco and D. Jin, “Onset of fermi degeneracy in a trapped atomic gas,” *Science*, vol. 285, no. 5434, pp. 1703–1706, 1999.
- [5] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, “Quantum phase transition from a superfluid to a mott insulator in a gas of ultracold atoms,” *Nature*, vol. 415, no. 6867, pp. 39–44, 2002.
- [6] M. Endres, T. Fukuhara, D. Pekker, M. Cheneau, P. Schauß, C. Gross, E. Demler, S. Kuhr, and I. Bloch, “The/higgs/’amplitude mode at the two-dimensional superfluid/mott insulator transition,” *Nature*, vol. 487, no. 7408, pp. 454–458, 2012.
- [7] R. Dubessy, T. Liennard, P. Pedri, and H. Perrin, “Critical rotation of an annular superfluid bose gas,” 04 2012. [Online]. Available: <http://fr.arxiv.org/abs/1204.6183>
- [8] T. Giamarchi, “Quantum physics in one dimension,” *J. Phys. A: Math. Gen*, vol. 37, pp. 5275–5276, 2004.
- [9] L. Landau, “The theory of a fermi liquid,” *Sov. Phys. JETP*, vol. 3, no. 6, p. 920, 1957.
- [10] P. Mathieu and D. Senechal, *Conformal field theory*. New York: Springer, 1997.
- [11] A. S. Sørensen, E. Demler, and M. D. Lukin, “Fractional quantum hall states of atoms in optical lattices,” *Phys. Rev. Lett.*, vol. 94, p. 086803, Mar 2005. [Online]. Available: <http://link.aps.org/doi/10.1103/PhysRevLett.94.086803>
- [12] L.-M. Duan, E. Demler, and M. D. Lukin, “Controlling spin exchange interactions of ultracold atoms in optical lattices,” *Phys. Rev. Lett.*, vol. 91, p. 090402, Aug 2003. [Online]. Available: <http://link.aps.org/doi/10.1103/PhysRevLett.91.090402>
- [13] E. Jané, G. Vidal, W. Dür, P. Zoller, and J. I. Cirac, “Simulation of quantum dynamics with quantum optical systems,” *Quantum Information & Computation*, vol. 3, no. 1, pp. 15–37, 2003.
- [14] T.-L. Ho and S. Yip, “Pairing of fermions with arbitrary spin,” *Physical review letters*, vol. 82, no. 2, pp. 247–250, 1999.
- [15] S. Capponi, G. Roux, P. Lecheminant, P. Azaria, E. Boulat, and S. R. White, “Molecular superfluid phase in systems of one-dimensional multicomponent fermionic cold atoms,” *Phys. Rev. A*, vol. 77, no. 1, p. 013624, Jan. 2008.
- [16] T. Fukuhara, S. Sugawa, Y. Takasu, and Y. Takahashi, “All-optical formation of quantum degenerate mixtures,” *Phys. Rev. A*, vol. 79, p. 021601, Feb 2009. [Online]. Available: <http://link.aps.org/doi/10.1103/PhysRevA.79.021601>

- [17] S. Taie, Y. Takasu, S. Sugawa, R. Yamazaki, T. Tsujimoto, R. Murakami, and Y. Takahashi, “Realization of a $SU(2) \times SU(6)$ system of fermions in a cold atomic gas,” *Phys. Rev. Lett.*, vol. 105, p. 190401, Nov 2010. [Online]. Available: <http://link.aps.org/doi/10.1103/PhysRevLett.105.190401>
- [18] T.-L. Ho and S. Yip, “Pairing of fermions with arbitrary spin,” *Phys. Rev. Lett.*, vol. 82, pp. 247–250, Jan 1999. [Online]. Available: <http://link.aps.org/doi/10.1103/PhysRevLett.82.247>
- [19] T. Stöferle, H. Moritz, C. Schori, M. Köhl, and T. Esslinger, “Transition from a strongly interacting 1d superfluid to a mott insulator,” *Phys. Rev. Lett.*, vol. 92, p. 130403, Mar 2004. [Online]. Available: <http://link.aps.org/doi/10.1103/PhysRevLett.92.130403>
- [20] R. Citro, A. Minguzzi, and F. Hekking, “Quantum stirring as a sensitive probe of 1d superfluidity,” 02 2009. [Online]. Available: <http://arxiv.org/abs/0902.1231>
- [21] J. Cardy, *Scaling and renormalization in statistical physics*. Cambridge University Press, 1996, vol. 5.
- [22] A. Kamenev and A. Levchenko, “Keldysh technique and non-linear σ -model: basic principles and applications,” *Advances in Physics*, vol. 58, pp. 197–319, May 2009.
- [23] H. Nonne, “Phases isolantes de mott des atomes froids fermioniques unidimensionnels à plusieurs composantes,” Ph.D. dissertation, Université de Cergy-Pontoise Laboratoire de Physique Théorique et Modélisation, Septembrer 2011.
- [24] H. J. Schulz, “Correlation exponents and the metal-insulator transition in the one-dimensional Hubbard model,” *Physical Review Letters*, vol. 64, pp. 2831–2834, Jun. 1990.
- [25] S. Lukyanov and A. Zamolodchikov, “Exact expectation values of local fields in the quantum sine-gordon model,” *Nuclear Physics B*, vol. 493, no. 3, pp. 571–587, 1997.
- [26] A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and strongly correlated systems*. Cambridge University Press, 2004.